STEADY-STATE THERMAL WAVES IN THE FREE-STREAM FLOW OF AN IDEAL GAS WITH A TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

K. B. Pavlov

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The structure of the stationary transition layer in a cold ideal gas impinging on a heated permeable surface (lattice) is investigated. Hysteresis occurs in establishing the structure of the stationary transition layer.

A characteristic feature of the propagation of heat in incompressible isotropic media with a thermal conductivity depending on the temperature T is the formation of temperature transition layers of the type of thermal waves with fronts which separate the regions with $\nabla T = 0$ and $\nabla T \neq 0$, [1, 2]. It is known [3, 4] that thermal waves localized in space can be detected also in the study of the heating of compressible media.

It is convenient to investigate the structures of temperature transition layers of the thermal waves type by considering steady-state heat transfer. Such considerations give a qualitative representation of separate stages of corresponding nonstationary regimes. Steady-state thermal waves in incompressible media with a temperature-dependent thermal conductivity or with heat sinks were discussed in [5-8]; in the following discussion we investigate steady-state transition layers of the thermal waves type in an ideal gas with the equation of state

$$p = A\rho T. \tag{1}$$

Here p and ρ are the pressure and density of the gas, $A \equiv R/\mu$, R is the universal gas constant, and μ is the mass of a kilomole of the gas.

Suppose an ideal gas (1) with a temperature-dependent thermal conductivity moves uniformly in the direction of the x axis through a permeable plane surface ("lattice") at $x = x_W$ whose temperature is $T = T_W$. If the velocity and density of the gas at $x = -\infty$ are $u = u_0$ and $\rho = \rho_0$, where u_0 and ρ_0 are positive constants, and $T = T_0 = 0$, the state of the gas in the transition layer for $x < x_W$ must be determined from (1) and the integrals

$$m\left(\frac{u^2}{2} + c_v T + \frac{p}{\rho}\right) - \varkappa \frac{dT^n}{dx} = \frac{I^2}{2m}, \quad mu + p = I, \tag{2}$$

$$\rho u = m, \quad I,\ m,\ c_v,\ \varkappa - \text{const} > 0, \quad n = \text{const} > 1,$$

obtained by integrating the equations of steady-state gasdynamics [2]. In the transition layer where $dT/dx \neq 0$ the temperature of the gas is everywhere continuous; this follows directly by integrating the first of Eqs. (2) with respect to x from $\Sigma - \varepsilon$ to $\Sigma + \varepsilon$, where $x = \Sigma$ is the surface of possible discontinuity of the solutions of system (1) and (2), and then letting $\varepsilon \to 0$. Therefore, in the case under consideration only isothermal jumps can be formed in the transition layer in which the gas parameters p, u, and ρ experience nonremovable discontinuities and T, a removable discontinuity. An isothermal jump will be called outer or inner depending on whether it occurs on the boundary or within the transition layer.

An investigation of the solution of the equation

$$\frac{dT^n}{dx} = \frac{(\gamma + 1)\, mc_v}{4\varkappa} T, \quad \gamma = \frac{c_p}{c_v}, \tag{3}$$

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which follows from (1) and (2) on the assumption that $T \ll T_M \equiv I^2/4m^2A$, shows that, in principle, a spatially localized transition layer of the thermal-wave type can be formed with a stationary front $x = x_\varphi$, $-\infty < x_\varphi < x_w$. By integrating (3) and using the condition $T(-\infty) = 0$ expressions for the steady-state temperature distribution on both sides of the thermal wave-front surface can be obtained,

$$T(x) = \begin{cases} \left[\frac{(\gamma+1)(n-1)\,mc_v}{4\varkappa n}\right]^{\frac{1}{n-1}} x^{\frac{1}{n-1}} & \text{for} \quad x \geqslant x_\varphi \\ 0 & \text{for} \quad x \leqslant x_\varphi . \end{cases}$$

From now on we assume the origin is coincident with the thermal wave-front surface $x_\varphi \equiv 0$. In the approximation assumed the distributions of u, ρ, and p near the thermal wave-front surface can also be found.

It is advantageous to represent the gas parameters $x_\varphi \le x \le x_w$ over the whole transition layer as functions of $z = z(x) = mu/p$, $0 \le z < \infty$. Then it follows from (1) and the last two of the integrals (2) that

$$p = \frac{I}{z+1}, \quad u = \frac{I}{m}\frac{z}{z+1}, \quad \rho = \frac{m^2}{I}\frac{z+1}{z}, \quad T = \frac{I^2}{m^2A}\frac{z}{(z+1)^2}. \tag{4}$$

The maximum value $T = T_M$ is reached at $z = 1$. Thus, in the steady flow of an ideal gas with definite I and m the admissible values of the temperature have an upper bound $T(z) \le T_M$. If $T_i < T_M$ the value of T_i can be attained for two values,

$$z_{i1,2} = \frac{2 - \tau_i \pm 2\sqrt{1-\tau_i}}{\tau_i}, \quad z_{i1}\cdot z_{i2} = 1, \tag{5}$$

$$\tau_i \equiv \frac{T_i}{T_M}.$$

Since

$$\frac{dT^n}{dx} = \frac{dT^n}{dz}\cdot\frac{dz}{dx}, \tag{6}$$

it follows from (2) and (4) that

$$(z - z_f)\frac{dx}{dz} = B_n \frac{z^{n-1}(1-z)}{(z+1)^{2n-1}}, \tag{7}$$

$$B_n \equiv \frac{\varkappa n(\gamma-1)(4T_M)^{n-1}}{mA}, \quad z_f \equiv \frac{\gamma-1}{2},$$

whose solution is $x = x(z)$. Solving this for $z = z(x)$ by using (4), the distributions of p, u, ρ, and T inside the transition layer can then be obtained. A particular solution of Eq. (7) which passes through the point (x_k, z_k) in the xz plane has the form

$$x - x_k = B_n \int_{z_k}^{z} \frac{\zeta^{n-1}(1-\zeta)\,d\zeta}{(\zeta+1)^{2n-1}(\zeta - z_f)} \equiv B_n \int_{z_k}^{z} \psi(\zeta)\,d\zeta, \tag{8}$$

which can be evaluated in terms of elementary functions for certain values of n and γ. For example, if $n = 2$, the solution of (8) has the form

$$x - x_k = B_2 \frac{\gamma-1}{(\gamma+1)^3}\left\{(\gamma-1)(3-\gamma)\ln\left|\frac{1-\xi}{1-\xi_k}\right| + 2\left[\xi(3\gamma-1-2\xi) - \xi_k(3\gamma-1-2\xi_k)\right]\right\}, \tag{9}$$

$$\xi \equiv \frac{\gamma+1}{2(z+1)}, \quad \xi_k \equiv \frac{\gamma+1}{2(z_k+1)}.$$

The integral curves (8) and (9) are qualitatively different for $\gamma > 3$, $\gamma = 3$, and $1 < \gamma < 3$. Figures 1-3 show families of integral curves (9) for $\gamma = 2$, 3, and 4; the solid lines are integral curves passing through the point (x_φ, z_φ), where $z_\varphi = z(x_\varphi) = \infty$, i.e., satisfying the condition on the front of the thermal wave. If $\gamma \ne 3$, the straight line $z = z_f$, which is a particular solution of the inverted Eq. (7) [9], is added to the family of solutions of Eq. (7).

By using (4), (6), and (7) an expression can be determined for the heat flux $q \equiv -\varkappa(dT^n/dx)$ as a function of z,

$$q = \frac{I^2}{m(\gamma-1)}\frac{z_f - z}{(z+1)^2}.$$

It follows from this that the flux is directed along the positive x axis for $z < z_f$, and along the negative x axis for $z > z_f$. At the points $z = z_\varphi$ and $z = z_f$ $q(z_\varphi) = q(z_f) = 0$; these points correspond to a uniform state of the gas with constant values of p, u, ρ, and T ahead of the front, and a stationary shock wave propagating

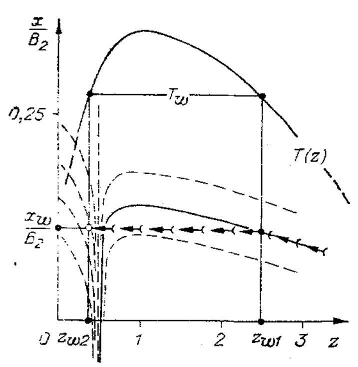

Fig. 1

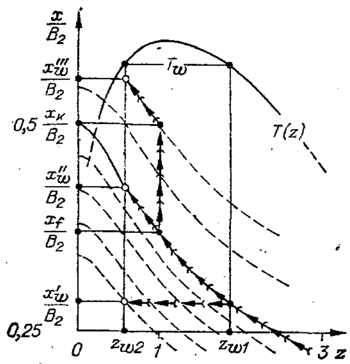

Fig. 2

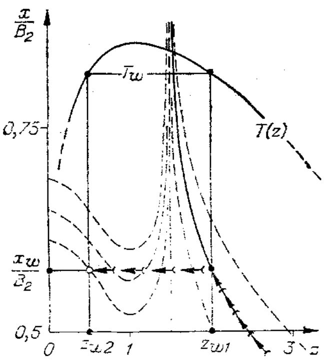

Fig. 3

in an ideal gas at temperature T_0 behind the front. It is obvious that the determination of the gas parameters in the transition layer with the lattice temperature $T_w = T(z_f) = T(z_f^{-1}) \equiv T_M [8(\gamma-1)/(\gamma+1)^2]$ represents a problem of the structure of a shock wave in a cold ideal gas with a temperature-dependent thermal conductivity.

According to (5) the temperature of the heated lattice T_w, which is less than T_M, occurs at two values of z: z_{w1} and z_{w2}. By applying the general condition of thermodynamic equilibrium to the system consisting of a single small area of the lattice – considered as a thermostat – and a column of gas passing through this area it can be determined which of the two conditions is more stable thermodynamically. Actually, since the increase in the total entropy of the thermodynamic system per unit time can be written in the form

$$\Delta S_{1,2} = m[s(z_{w1,2}) - s(z_\varphi)] + \frac{q}{T}(z_{w1,2}),$$

where

$$s(z) = s_0 + c_v \ln \frac{z^\gamma}{(z+1)^{\gamma+1}}, \quad s_0 = \text{const},$$

by using (5) and (7) it follows that for any values of γ for $T_w < T_M$, $\Delta S_1 < \Delta S_2$. Thus, the state with z_{w2} is thermodynamically more stable than the state with z_{w1}.

The change in gas parameters in the transition layer directly behind the thermal wave front $x = x_\varphi$ corresponds to the motion of a representative point on the integral curve $x = x(z)$ (8) with $x_k \equiv x_\varphi$ and $z_k \equiv z_\varphi$. During the succeeding stages this motion can proceed continuously or with a jump related to the transition of the representative point to another integral curve. Any displacement of the representative point in the xz plane must occur with $\Delta x \geq 0$, and the transition from one integral curve to another must be isothermal (Figs. 1-3).

We consider the possible displacements of a representative point from the initial state (x_φ, z_φ) to the final state (x_w, z_{w2}). The solid arrows denote continuous motion of the representative point along an integral curve, and the broken arrows indicate a jump to another integral curve.

We begin with the case $1 < \gamma < 3$.

If $T_w = T(z_f)$, the representative point is displaced according to the scheme

$$(x_\varphi, z_\varphi) \to (x_f, z_f^{-1}) \dashrightarrow (x_f, z_f) \to (x_w, z_f),$$

$$x_f = B_n \int_{z_\varphi}^{z_f^{-1}} \psi(\zeta)\, d\zeta \quad x_w = \text{const} \geqslant x_f \quad z_f \equiv z_{w2}.$$

Since an isothermal jump transfers the gas to a state in which there is no interaction of the lattice with the gas, the lattice can be at any distance from the isothermal jump $(x_w \geq x_f)$, and the value of x_w is not determined. The isothermal jump is located outside the transition layer in which $dT/dx > 0$ everywhere.

If $T_w \equiv T(z_{w1,2}) = T(z_f)$, the displacement of the representative point is described by the scheme

$$(x_\varphi, \to (x_w, z_{w1}) - - \to (x_w, z_{w2}), \tag{10}$$

$$x_w = B_n \int_{z_\varphi}^{z_{w1}} \psi(\zeta)\, d\zeta$$

with the outer isothermal jump located directly on the lattice (Fig. 1), or according to the scheme

$$(x_\varphi, z_\varphi) \to (x_k, z_{k1}) - - \to (x_k, z_{k2}) \to (x_w, z_{w2}), \tag{11}$$

$$x_k = B_n \int_{z_\varphi}^{z_{k1}} \psi(\zeta)\, d\zeta, \quad x_w = x_k + B_n \int_{z_{k2}}^{z_{w2}} \psi(\zeta)\, d\zeta$$

with an inner isothermal jump. When the representative point is displaced according to scheme (11), if $T_w > T(z_f)$, $T_k \equiv T(z_{k_{1,2}}) > T(z_f)$, $z_{k2} > z_f$ and $dT/dx > 0$ everywhere within the transition layer $x_\varphi < x < x_w$ (Fig. 4a); if $T_w < T(z_f)$, $T_k < T(z_f)$, $z_{k2} < z_f$, $dT/dx > 0$ for $x_\varphi < z < z_k$, and $dT/dx < 0$ for $x_k < x < x_w$ (Fig. 4b); i.e., at an isothermal jump the temperature of the gas has its maximum value within the transition layer.

We consider next the case $\gamma = 3$. If $T(z_f) = T_M$, $T_w \leq T(z_f)$, the particular solutions (8) satisfy Eq. (7), where there is a cancellation of $\zeta - 1$ by $\zeta - z_f$, and also the root $z = z_f = 1$, which corresponds to a vertical straight line intersecting the integral curves (8) in the xz plane (Fig. 2). In considering possible displacement schemes of the representative point there should be kept in mind the possibility of its continuous transition from one integral curve to another along the straight line $z = 1$. The motion of the representative point along the straight line $z = z_f = 1$ in this case corresponds to a uniform state of the gas with constant values of p, u, ρ, and $T = T_M$. Therefore, a continuous transition from the integral curve passing through the point (x_φ, z_φ) along the straight line $z = 1$ means a continuous change of the gas parameters within a transition layer of finite thickness from constant values at a temperature zero to constant values at a temperature different from zero (a transition layer without a jump localized in space with leading and trailing edges of the thermal-wave type). The formation of a thermal wave front is related to the existence of a singular solution of the corresponding ordinary differential equation [8]. We show that a similar situation occurs for a transition layer without a jump.

From the first integral of (2) and (4) and (5) a differential equation can be obtained describing the temperature distribution within the transition layer,

$$\frac{\varkappa T_M^{\,n-1}}{mc_v}\frac{d\omega}{dx} = \mathcal{F}_\gamma(\omega) \equiv \frac{2 - \omega^{\frac{1}{n}}(1 + z_f) + 2\sqrt{1 - \omega^{\frac{1}{n}}}}{2 - \omega^{\frac{1}{n}} + 2\sqrt{1 - \omega^{\frac{1}{n}}}}\,\omega^{\frac{1}{n}} \quad \omega \equiv \tau^n. \tag{12}$$

For $\gamma = 3$ the solutions of (12) are $\omega = 0$ and $\omega = 1$ ($T = 0$ and $T = T_M$). The necessary and sufficient criterion for $\omega = 0$ and $\omega = 1$ to be singular solutions of Eq. (12) is the convergence of the improper integral $\int_0^1 \frac{d\omega}{\mathcal{F}_3(\omega)}$ [9], in which the singularities $\omega = 0$ and $\omega = 1$ are integrable. For a transition layer without a jump the singular solution of Eq. (12) makes it possible to represent the temperature distribution in space by joining two singular solutions and one particular solution. For $\gamma \neq 3$ Eq. (12) has one singular solution $\omega = 0$, and therefore, a spatially localized transition layer without a jump is impossible.

We continue the discussion of possible displacement schemes for the representative point when $\gamma = 3$.

If $T_w = T(z_f)$, the representative point is displaced according to the scheme

$$(x_\varphi, z_\varphi) \to (x_f, z_f) \to (x_w, z_f),$$

$$x_f = B_n \int_{z_\varphi}^{z_f} \psi(\zeta)\, d\zeta \quad x_w = \text{const} \geqslant x_f \quad z_f \equiv z_{w2}.$$

In this case the gas parameters vary continuously inside a transition layer of finite thickness which can be located at any distance from the lattice.

If $T_w < T(z_f)$, the representative point is displaced according to scheme (10) with an outer isothermal jump or according to scheme (11) with an inner jump at which the temperature has its maximum value inside the transition layer. In this case a continuous transition is also possible according to the scheme

$$(x_\varphi, z_\varphi) \to (x_f, z_f) \to (x_k, z_f) \to (x_w''', z_{w2}),$$

$$x_f = B_n \int_{z_\varphi}^{z_f} \psi(\zeta)\, d\zeta \quad x_k = \text{const} > x_f$$

$$x_w''' = x_k + B_n \int_{z_f}^{z_{w2}} \psi(\zeta)\, d\zeta$$

(Fig. 4b).

We consider the case $\gamma > 3$, $T_W \le T(z_f)$. If $T_W = T(z_f)$, the representative point moves in the xz plane along an integral curve of Eq. (8) from $x_k \equiv x_\varphi$ and $z_k \equiv z_\varphi$ to the infinitely distant point ($x_W = \infty$, z_f); this motion corresponds to a semi-infinite transition layer with continuously varying gas parameters. If $T_W < T(z_f)$, the possible displacements of the representative point are similar to the corresponding cases with an outer or inner isothermal jump for $\gamma = 3$ (Fig. 3).

By taking account of the schemes discussed for the displacement of the representative point and using Eqs. (4) and (8) the values of the gas parameters within the transition layer can be calculated completely. The qualitative character of the change of temperature T and velocity u of the transition layer corresponding to scheme (11) can be seen in Fig. 4. It may seem strange that for the same lattice temperature T_W the transition layer can have a different structure. This is related to the absence of a unique solution of Eq. (7) and indicates the distinctive hysteresis which occurs in establishing a steady-state transition layer. To demonstrate the lack of uniqueness of the structure of the transition layer it is necessary to examine further the stability of a transition layer with an isothermal jump anywhere within it. A detailed discussion of the case described by Eqs. (3) and (5) is rather tedious. For a theoretical solution of the problem of the uniqueness of the structure of a transition layer with an isothermal jump we can restrict ourselves to an investigation of a transition layer arising for the isothermal equation of state

$$p = a^2\rho, \quad a = \text{const} \tag{13}$$

and a linear thermal conductivity [n = 1 in the first integral of (2)]. It follows from (7) and the last two integrals of (2) that the density, pressure, and velocity of the gas in front of an isothermal jump and behind it are constant and equal, respectively, to

$$\rho_0,\ p_0 = a^2\rho_0,\ u_0;\ \rho_f \equiv \rho_0 \frac{u_0^2}{a^2}, \qquad p_f = a^2\rho_f,\ \ u_f \equiv u_0 \frac{a^2}{u_0^2}. \tag{14}$$

Assuming $u_0 > a$ we note that Eqs. (14) hold independently of the temperature distribution in the transition layer found from the first of integrals (2). If the isothermal jump is located at x = 0, and $u(-\infty) = u_0$, the temperature distribution can be written in the form

$$T(x) = \begin{cases} T_0\left(1 - e^{\frac{mc_p}{\varkappa}x}\right) + T_s e^{\frac{mc_p}{\varkappa}x} & \text{for } -\infty < x \le 0 \\ \left(T_0 + \frac{u_0^2 - u_f^2}{2c_p}\right)\left[1 - e^{\frac{mc_p}{\varkappa}(x - x_w)}\right] + T_w e^{\frac{mc_p}{\varkappa}(x - x_w)} & \text{for } 0 \le x \le x_w, \end{cases} \tag{15}$$

where T_S is the temperature of the gas at the isothermal jump. The unknown constants x_W and T_S cannot be determined simultaneously from the equation

$$T_s = \left(T_0 + \frac{u_0^2 - u_f^2}{2c_p}\right)\left(1 - e^{-\frac{mc_p}{\varkappa}x_w}\right) + T_w e^{-\frac{mc_p}{\varkappa}x_w} \tag{16}$$

This gives rise to the ambiguous character of the structure of the transition layer (the location of the isothermal jump relative to the lattice is not uniquely determined). Actually, for $T_W < T_0 + [(u_0^2 - u_f^2)/2c_p]$ the isothermal jump is located arbitrarily within the transition layer $-\infty < x \le x_W$; for $T_W = T_0 + [(u_0^2 - u_f^2)/2c_p]$ $T_S = T_W$, but x_W can be arbitrary.

The stability of the structure of the transition layer (13)-(16) is investigated under the assumption that the isothermal jump is subjected to an infinitesimal displacement in a direction perpendicular to its plane. This is done by writing all quantities in the nonstationary system of one-dimensional gasdynamics equations [2] as sums of the stready-state values (14) and (15) and infinitesimal nonstationary perturbations.

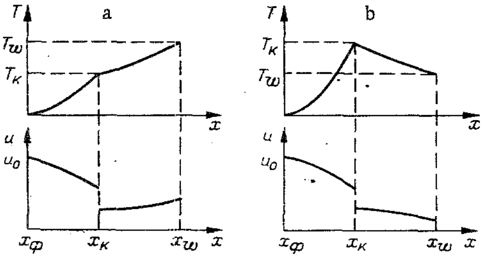

Fig. 4

After performing the usual linearization we have for the perturbations of the density and velocity

$$\frac{\partial \rho_i'}{\partial t} + u_i \frac{\partial \rho_i'}{\partial x} + \rho_i \frac{\partial u_i'}{\partial x} = 0,$$

$$\frac{\partial u_i'}{\partial t} + u_i \frac{\partial u_i'}{\partial x} + \frac{a^2}{\rho_i} \frac{\partial \rho_i'}{\partial x} = 0.$$

Here primes denote infinitesimal perturbations; for quantities in front of the isothermal jump $i = 0$, and for those behind the jump $i = f$. In investigating the stability we seek solutions for ρ_i' and u_i' in (15) which are proportional to $e^{\Omega t}$, $\Omega > 0$. Taking account of the continuity conditions for mass and momentum fluxes at the isothermal jump, linearized with respect to small perturbations, and the fact that $\rho_i', u_i'(x) \to 0$ as $x \to \pm\infty$, it is easy to see that for $\Omega > 0$ ρ_i' and $u_i'(x) \equiv 0$. However, from this there still does not follow the stability of the structure of a transition layer with an isothermal jump located arbitrarily within it, since the behavior of the perturbation T_i' remains unexplained. By linearizing the nonstationary energy equation [2] and taking account of the fact that ρ_i', $u_i' \equiv 0$, the following equations can be obtained for the temperature perturbation:

$$\frac{\partial T_i'}{\partial t} + \gamma u_i \frac{\partial T_i'}{\partial x} = \chi_i \frac{\partial^2 T_i'}{\partial x^2}, \quad \chi_i \equiv \frac{\varkappa}{\rho_i c_v},$$

whose solutions, increasing with time, must also be sought in the form $\bar{T}_i(x)e^{\Omega t}$ with $\Omega > 0$.

It is obvious that $\bar{T}_0'(x) \equiv 0$ if $x_w = 0$.

If $x_w > 0$,

$$\bar{T}_0' = C_1 e^{k_1^+ x}, \quad \bar{T}_f' = C_2 e^{k_2^+ x} + C_3 e^{k_2^- x},$$

$$k_i^{+-} = \frac{\gamma u_i \pm \sqrt{\gamma^2 u_i^2 + 4\chi_i \Omega}}{2\Omega},$$

since $\bar{T}_0'(-\infty) = 0$. Taking account of the fact that $\bar{T}_f'(x_w) = 0$ and that the temperature and energy flux perturbations are continuous at $x = 0$, it is easy to see that C_1, C_2, $C_3 \equiv 0$, and therefore, $T_i' \equiv 0$ for $-\infty < x \le x_w$.

Thus, a stationary transition layer in a cold gas impinging on a heated permeable surface (lattice) may actually have a nonunique structure.

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